Public Abandonment of Small Change: a Stability Analysis

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Abstract

This study demonstrates the role of the carrying cost of money in modeling the public abandonment of small denomination. With a small positive carrying cost, there are two types of equilibrium paths that lead to a steady state in which the smallest denomination does not circulate, representing discarding pennies and converting pennies into larger coins, respectively. We also demonstrate that two kinds of steady states with a full-support wealth distribution exist and that they are both locally stable and determinate.

(JEL classification: C62, C78, E40)

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1 Introduction

This note models the public abandonment of small denomination such as pennies. By public abandonment, we mean the following; initially some people carry pennies but eventually nobody does. We consider two forms of such abandonment: discarding and conversion into a larger denomination. In the United States, pennies are sometimes discarded by individuals. If everyone does so, pennies will disappear from the economy. This is the first type of public abandonment. On the other hand, even if people do not

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discard pennies, if they convert pennies into larger denomination whenever there is chance to do so and never break larger units into pennies, pennies will eventually cease to circulate. This is the second type of public abandonment. In such processes, prices must be rounded up or down so the trade can be done without pennies.

Lee et al. (2005) [3] is the first to model denomination of currency and the trade-off between carrying cost and flexible transaction that small denomination has. A suitable framework to model that trade-off is a model in which money is essential and indivisible. Lee et al. (2005) [3] therefore builds their model on the Zhu (2003) model, which in turn is a Trejos-Wright (1995) random matching model extended to a larger wealth set. At each date, there is a portfolio choice stage in which individuals choose in what form they should hold their wealth (e.g., one \$5 vs. five \$1's), followed by a standard matching and trading stage. In such an environment they show existence of steady states. We use their framework to study when and how public abandonment can take place. Our study is essentially a stability analysis in their model; we show existence of dynamic equilibria in which the economy converges to steady states. In some of the dynamic equilibria, the smallest denomination ceases to circulate: public abandonment.

Lee et. al. allows for an arbitrary bound on wealth. We provide an analysis of stability in the simplest possible case; the bound is two. In this economy, people's wealth is in set $\{0, 1, 2\}$ and there are only two denominations, \$1 bill and \$2 bill, so \$1 bills in our model correspond to 'pennies' in the real world. Despite the simplicity, this setup has all the essence of Lee et. al. In particular, when an individual who has wealth of 2 chooses his portfolio, he experiences the trade-off or dilemma of denomination. That is, while choosing one \$2 incurs less carrying cost than choosing two \$1's, this may lead to losing a trading opportunity; if he meets a seller without a \$1 bill, he will not have an option of spending only \$1 because the seller cannot offer change. We show that in such an environment both types of public abandonment are possible. More precisely, it is shown that there exist two types of non-full-support steady states, steady states in which everyone's wealth is in $\{0, 2\}$. The economy can jump to one of them by means of discarding. The economy can also gradually converge to the other by means of conversion. This is in sharp contrast to the case of zero carrying cost, where the dilemma of denomination does not occur. Specifically, Huang and Igarashi (2014) [2] show that such gradual convergence is impossible for the non-full-support steady state in the Zhu (2003) model.

We also study the stability of full-support steady states, ones in which the wealth distribution has support $\{0, 1, 2\}$. There are two types of such steady states, one supported by a pure strategy and the other supported by a mixed strategy. In the pure-strategy steady state, all the wealth is held in the form of \$1 bills and nobody holds a \$2 bill. It serves as an example in which the largest unit (e.g. \$100 bill in the US) does not circulate. In the mixed-strategy steady state, both types of bills circulate. Because Lee et. al. (2005) does not answer whether the smallest unit and larger units can circulate at the same time in equilibrium, the above mixed-strategy steady state provides the first example in which a larger unit actually matters. Finally, it is shown that both types of full-support steady states are stable and determinate. To the best of our knowledge, our approach to the stability of the mixed-strategy steady state is new.

2 Model

The model is identical to Lee et al.'s [3] so we refer readers to their sections 2 and 3 for details. The present section provides an outline of the model and introduces some notations. We normalize the smallest denomination of money to be one and let $m \in (0, 1)$ denote the per capita wealth divided by the bound on individual wealth Z.

The model has two stages at each date: the 'portfolio choice stage' followed by the 'pairwise matching stage'. In the portfolio choice stage, agents can choose a portfolio of monetary items within their wealth but at no cost. An individual portfolio is $y = (y_1, \cdots, y_K)$, where $y_k \geq 0$ is the integer quantity of the k-th smallest coin/bill held by the individual. Free discarding of money is allowed at this stage.

In the pairwise matching stage, agents become a buyer (so the partner is a seller) with probability $1/n$. With probability $1 - 2/n$, the match is a no-coincidence meeting. In single-coincidence meetings, the buyer makes a take-it-or-leave-it offer. This offer consists of the amount of production, the monetary items that the buyer should transfer, and the monetary items that the seller should transfer ("change"). Randomization is allowed both in the portfolio choice and in the buyer's take-it-or-leave-it offer.

The time discount factor is $\beta \in (0, 1)$ and the period utility is $u(c) - q - \gamma \sum_k y_k$, where $c \in \mathbb{R}_+$ is the amount of consumption, $q \in \mathbb{R}_+$ is the amount of production, and $\gamma \geq 0$ is the utility cost of carrying money of any size

from the portfolio choice stage to the pairwise stage. Function $u : \mathbb{R}_+ \to \mathbb{R}_+$ has all nice properties: $u(0) = 0$, $u'(\infty) = 0$, $u' > 0$, $u'' < 0$, and sufficiently large but finite $u'(0)$.

Let π^t and w^t be the distribution and the value function on wealth prior to the portfolio choice stage. Also, let θ^t and h^t be the distribution and the value function on portfolios prior to the pairwise matching stage. These distributions and value functions must satisfy all law of motions and Bellman equations in Lee et. al. When a non-stationary, dynamic equilibrium is considered, the initial condition π^0, the distribution of wealth prior to the portfolio stage at $t = 0$, is given. For distributions and value functions, we use superscripts for date and subscripts for states throughout the paper.

Definition 1 *Given π^0, an equilibrium is a sequence $\{(\theta^t, \pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies the law of motions and Bellman equations of [3]. A tuple (θ, π, w) is a monetary steady state if $(\theta^t, \pi^t, w^t) = (\theta, \pi, w)$ for $t \geq 0$ is an equilibrium and if $w \neq 0$. Pure-strategy steady states are ones that have unique optimal choices in the portfolio stage and in all pairwise meetings. Other steady states are called mixed-strategy steady states.*[1]

Next we define stability.

Definition 2 *A steady state (θ, π, w) is locally stable if there is a neighborhood of π such that for any initial distribution in the neighborhood, there is an equilibrium path such that $(\theta^t, \pi^t, w^t) \to (\theta, \pi, w)$. A locally stable steady state is determinate if for each initial distribution in this neighborhood, there is only one equilibrium that converges to it.*

The following involves two possible ways that the economy whose initial distribution is different from that of a steady state reaches the steady state. One is to jump to the steady state in one period, which possibly involves some people discarding some money at the initial date. The other is a gradual convergence that does not involve discarding of money.

3 Public Abandonment of Small Change

The economy we analyze has wealth bound $Z = 2$ so the per capita wealth is $2m$. Consequently, the wealth distribution is $\pi = (\pi_0, \pi_1, \pi_2)$ and the value of

[1] The last includes degenerate mixed strategies.

wealth at the beginning of the portfolio choice stage is $w = (w_0, w_1, w_2)$. The economy's denomination structure is {\$1 bill, \$2 bill}. There are four possible individual states at the beginning of the matching stage: $y \in Y \equiv$\{0, \$1, 2\$1s, \$2\}, where \$1 and \$2 represent holding of one \$1 bill and holding of one \$2 bill, respectively, and 2\$1s represents holding of two \$1 bills. Consequently, the distribution and the value function over these four states are $\theta = (\theta_0, \theta_1, \theta_{11}, \theta_2)$ and $h = (h_0, h_1, h_{11}, h_2)$, respectively, where the subscripts 1, 11, and 2 indicate \$1, 2\$1s, and \$2, respectively. The normalization $u(0) = 0$ and the buyer take-it-or-leave-it offer imply $w_0 = h_0 = 0$.

A (b, s)-match is a trade meeting between a buyer with state $b \in Y$ and a seller with state $s \in Y$. There are five kinds of trade meetings in which positive production can take place: (\$1, 0)-meeting, (\$1, \$1)-meeting, (\$2, \$1)-meeting, (\$2, 0)-meeting, and (2\$1s, 0)-meeting. Notice that in the (\$2, \$1)-meeting, the buyer can transfer exactly one dollar by the seller offering change. (2\$1s, \$1)-meeting and (\$2, \$1)-meeting share the same set of possible trading opportunities (monetary transfer in particular). But more possible trading opportunities are possible in (2\$1s, 0)-meeting than in (\$2, 0)-meeting. The benefit of holding two \$1 over a \$2 occurs only in The (2\$1s, 0)-meeting.

At all the steady states of our interest, $\theta_0 = \pi_0$, $\theta_1 = \pi_1$ and $\theta_{11} + \theta_2 = \pi_2$ hold. The Bellman equations are

$$\begin{aligned}
h_1 &= \frac{n-1+\pi_2}{n}\beta w_1 + \frac{\pi_0}{n}\max\{u(\beta w_1),\ \beta w_1\} \\
&\quad + \frac{\pi_1}{n}\max\{u(\beta w_2 - \beta w_1),\ \beta w_1\} - \gamma \\
h_{11} &= \frac{n-1+\pi_2}{n}\beta w_2 + \frac{\pi_0}{n}\max\{u(\beta w_2),\ u(\beta w_1) + \beta w_1,\ \beta w_2\} \\
&\quad + \frac{\pi_1}{n}\max\{u(\beta w_2 - \beta w_1) + \beta w_1,\ \beta w_2\} - 2\gamma \\
h_2 &= \frac{n-1+\pi_2}{n}\beta w_2 + \frac{\pi_0}{n}\max\{u(\beta w_2),\ \beta w_2\} \\
&\quad + \frac{\pi_1}{n}\max\{u(\beta w_2 - \beta w_1) + \beta w_1,\ \beta w_2\} - \gamma, \qquad (1)
\end{aligned}$$

where the max operators correspond to the optimization problem of trade, and

$$\begin{aligned}
w_1 &= \max\{h_1, 0\} \\
w_2 &= \max\{h_{11}, h_2, 0\}, \qquad (2)
\end{aligned}$$

where the max operators correspond to portfolio decision.

Four kinds of monetary steady states exist: two with a full-support distribution (i.e., $\pi_1 > 0$) and two with a non-full-support distribution (i.e., $\pi = (1-m, 0, m)$). The full-support steady states are discussed in the next section. The non-full-support steady states are the focus of our study because if the economy converges to them under some condition, that will be the model description of public abandonment of the smallest denomination. To describe the two non-full-support steady states, consider the following equation for $x \geq 0$, which is obtained by imposing $w_2 = h_2 = x$, $\pi_1 = 0$, and $\max\{u(\beta w_2), \beta w_2\} = u(\beta w_2)$ on the Bellman equations (1) and (2), namely

$$x = \frac{n+m-1}{n}\beta x + \frac{1-m}{n}u(\beta x) - \gamma. \tag{3}$$

As long as $u'(0)$ is large enough to satisfy the Trejos-Wright condition

$$u'(0) > \frac{n(1-\beta)}{\beta(1-m)} + 1,$$

for any sufficiently small $\gamma > 0$, equation (3) has two positive solutions, denoted $\bar{w} > \underline{w} > 0$. As γ approaches zero, $\underline{w}$ goes to zero, the non-full-support steady state value of holding one unit of money in the case of $\gamma = 0$. One can show $u(\bar{w}) > \bar{w}$ and $u(\underline{w}) > \underline{w}$.

Discarding

There is a non-full-support steady state such that $w_1 = 0$, $h_1 = -\gamma$, and $w_2 = h_2 = \bar{w}$. Notice that $h_2 > h_{11}$. Ones with wealth 2 choose to hold a \$2 bill. Because $h_1 < 0$, ones with wealth 1 discard the \$1 bill at the portfolio stage. In (\$2, 0)-meetings, a \$2 bill is transfered. In (\$1, \$1)-meetings and (\$2, \$1)-meetings, exactly one unit of wealth would be transferred for positive amount of production, but such meetings do not take place with positive probabilities because $\pi_1 = \theta_1 = 0$.

Consequently, if the initial distribution deviates from the steady state distribution $(1-m, 0, m)$, then those with only one \$1 bill choose to discard it at the initial date to avoid carrying cost. The economy 'jump' in one period to a non-full-support steady state with a lower stock of money $2m - \pi_1^0$.[2]

[2]This jump is possible even if $\gamma = 0$. However, when $\gamma = 0$, discarding money is a weakly dominated strategy. Also, the jump by discarding when $\gamma = 0$ would be eliminated by introducing even a tiny cost for discarding.

Conversion into larger bills

The other non-full-support steady state has $w_1 = h_1 = \underline{w}$ and $w_2 = h_2 = \bar{w}$. Everyone with wealth 2 chooses to hold a \$2 bill and \$2 is transferred in (\$2, 0)-meetings.[3] It is optimal to have one unit of monetary wealth being transferred for a positive amount of production in (\$1, 0)-meetings (resp. (\$2, \$1)-meetings), because $u(\beta\underline{w}) > \beta\underline{w}$ (resp. $u(\beta\bar{w}) > \beta\bar{w}$ and $0 < \beta w_2 - \beta w_1 < \beta\bar{w}$), although such meetings do not take place with positive probability at the steady state.

What would happen in (\$1, \$1)-meetings depends upon the size of γ. Under the Trejos-Wright condition, we can find unique $\gamma^* > 0$ such that (3) has only one solution. As γ moves from 0 to γ^*, $u(\beta w_2 - \beta w_1)$ decreases and βw_1 increases. The Intermediate Value Theorem then implies the existence of $\gamma' \in (0, \gamma^*)$ such that the two are equal. When $\gamma \in (0, \gamma')$, the strictly optimal strategy for the buyer in (\$1 \$1)-meetings is to transfer a \$1 bill. When $\gamma \in (\gamma', \gamma^*)$, zero payment is strictly optimal in (\$1 \$1)-meetings.

Although (\$1, \$1)-meetings do not occur with positive probabilities at the steady state, they are important because if the economy starts with $\pi_1^0 > 0$, then whether it converges to the above non-full-support steady state is dictated by whether \$1 is transferred in (\$1, \$1)-meetings.

If $\gamma \in (0, \gamma')$, a \$1 bill is transferred in (\$1, \$1)-meetings and then two \$1 bills are converted into one \$2 bill. This process has a peculiar property that we call *unit-root convergence*. That is, as π_1 approaches zero, the frequency of (\$1, \$1)-meetings goes to zero much faster, and as a result, the convergence becomes extremely slow in the end. Nevertheless, it can be shown that such convergent path to the steady state exists. For the proof, we derive a difference equation system of three variables (π_1^t, w_1^t, w_2^t) and study the three eigenvalues of its linearization around the steady state. It is shown that the stable manifold is two-dimensional.[4] Because the initial condition is one-dimensional (that is, there is only one initial condition π_1^0), there is a continuum of equilibrium paths converging to the steady state. It is concluded that this steady state is locally stable and indeterminate.[5] Note that this local stability is not so obvious. When $\gamma = 0$, such convergent

[3] If $u'(0)$ is large, $\bar{w}$ and $\underline{w}$ exist for even a large γ. In that case, $\frac{\pi_0}{n}u(\beta w_2) > \frac{\pi_0}{n}\{u(\beta w_1) + \beta w_1\} - \gamma$ holds so $h_2 > h_{11}$.

[4] Please find the matrix computation in the appendix.

[5] This indeterminacy has some resemblance to that of a non-monetary steady state of an overlapping generations model of fiat money with no carrying cost.

equilibrium does not exist. As proposition 5 of [2] implies, the non-full-support steady state becomes unstable when $\gamma = 0$.

If $\gamma \in (\gamma', \gamma^*)$, the value w_1 becomes so large near the steady state that reserving a \$1 bill is preferred in (\$1, \$1)-meetings. Neither discarding of \$1 bills nor gradual conversion into \$2 bills is optimal. The economy does not converge to this second type of non-full-support steady state.

These results are summarized in the following proposition. The detailed proof is in section 5.

Proposition 1 *There are two kinds of steady states in which \$1 bills do not circulate: one with $w_1 = 0$ and one with $w_1 > 0$. When the former exists, the economy can jump to it in one period by discarding of money, so it is locally stable and determinate. When the latter exists, it is locally stable (gradual convergence) and indeterminate if $\gamma \in (0, \gamma')$, and it is unstable if $\gamma \in \{0\} \cup (\gamma', \gamma^*)$.*

4 Full-support steady states

There are steady states in which public abandonment of small change does not happen, namely, steady states with a full-support wealth distribution. There are two such steady states: one with a pure strategy and one with a mixed strategy. Tables 1 and 2 show the equilibrium strategies, what bill agents choose at the portfolio choice stage and how much the buyer offers to the seller at the pairwise stage. In the mixed strategy, randomization occurs in the portfolio choice stage (i.e., $h_{11} = h_2$) but not in the buyer's offer. That is, agents with wealth 2 randomize over two \$1 bills and one \$2 bill in portfolio stage. Those who chose to hold one \$2 bill offer \$2 in a (\$2, 0)-meeting and those who chose to hold two \$1 bills offer only \$1 in a (2\$1s, 0)-meeting.

Proposition 2 *Both pure- and mixed-strategy steady states with a full-support wealth distribution exist generically and both are locally stable and determinate. Moreover, the convergence to the pure-strategy steady state is gradual convergence while that to the mixed-strategy steady state is by means of a jump in one period. Neither convergence involves discarding.*

The pure-strategy steady state exists for β sufficiently close to one, while the mixed-strategy steady state exists for β of intermediate size. In the

Table 1: Pure Strategy

Portfolio stage	Always choose \$1 bills				
			Seller's wealth		
			0	1	2
Pairwise stage	Buyer's wealth	0	-	-	-
		1	\$1	\$1	-
		2	\$1	\$1	-

Table 2: Mixed Strategy

Portfolio stage	Randomize 2 \$1's and 1 \$2 if wealth is 2; choose \$1 otherwise					
				Seller's wealth		
				0	1	2
Pairwise stage	Buyer's wealth	0	-	-	-	-
		1	-	\$1	\$1	-
		2	2 \$1 bills	\$1	\$1	-
			1 \$2 bill	\$2	\$1	-

pure-strategy steady state, \$2 bills do not circulate, so it serves as a model description of the circumstances that the largest bill (e.g., the US \$100 bill) does not circulate. Our mixed-strategy steady state serves as an example in which small bills and large bills co-exist.[6]

The proof for the stability of the pure-strategy steady state is standard. We derive a difference equation system of three variables (π_1^t, w_1^t, w_2^t), derive the three eigenvalues of the Jacobian at the steady state, and show that the stable manifold is one-dimensional. (Because we have one initial condition, this implies a unique path.) For the mixed-strategy steady state, it is shown that if the initial distribution is sufficiently close to that of the steady state, the economy can jump to the steady state by choosing appropriate random-

[6]While Lee et. al. use a fixed point theorem to show the existence of a steady state in a model with a denomination structure, they do not tell what bills are actually circulating in that steady state.

ization in the initial portfolio stage. To ensure that this convergence by jump is the only equilibrium path to the mixed-strategy steady state, we rule out the possibility of gradual convergence. Such a convergent path would have to preserve the indifference condition for the mixed strategy all along the path. By applying z-transform to the dynamical system, we show that preserving the indifference is generically not possible.

5 Proof outlines

Express π_0^t and π_2^t in terms of π_1^t using $\sum_{i=0}^2 \pi_i^t = 1$ and $\sum_{i=0}^2 i\pi_i^t = 2m$:

$$\begin{aligned} (\pi_0^t, \pi_2^t) &= (1 - m - \frac{\pi_1^t}{2}, m - \frac{\pi_1^t}{2}) \\ \text{where } \pi_1^t &\in \Pi \equiv [0, 2\min\{m, 1-m\}]. \end{aligned} \tag{4}$$

Let κ^t be the probability of paying \$1 in the (\$1, \$1)-meetings, and η^t the probability that an agent with wealth 2 chooses two \$1 bills at the portfolio stage in period t. Then the law of motion is

$$\pi_1^{t+1} = \pi_1^t - \frac{2(\pi_1^t)^2}{n}\kappa^t + \frac{2}{n}\left(1 - m - \frac{\pi_1^t}{2}\right)\left(m - \frac{\pi_1^t}{2}\right)\eta^t \tag{5}$$

Under the conjecture that (i) discarding of money does not occur, that (ii) \$1 is transfered in (\$1, 0)-meetings and in (\$2, \$1)-meetings, and that (iii) a positive amount of money is transfered in (\$2, 0)-meetings, the Bellman equation, defined on $W \equiv \{(w_1, w_2) | 0 \le w_1 \le w_2\}$, becomes as follows :

$$\begin{aligned} w_1^t &= \frac{n-1+\pi_2^t}{n}\beta w_1^{t+1} + \frac{\pi_0^t}{n}u(\beta w_1^{t+1}) \\ &\quad + \frac{\pi_1^t}{n}\max[u(\beta w_2^{t+1} - \beta w_1^{t+1}), \beta w_1^{t+1}] - \gamma \qquad (6) \\ w_2^t &= \frac{n-1+\pi_2^t}{n}\beta w_2^{t+1} + \max[\frac{\pi_0^t}{n}[u(\beta w_1^{t+1}) + \beta w_1^{t+1}] - \gamma, \frac{\pi_0^t}{n}u(\beta w_2^{t+1})] \\ &\quad + \frac{\pi_1^t}{n}[u(\beta w_2^{t+1} - \beta w_1^{t+1}) + \beta w_1^{t+1}] - \gamma. \qquad (7) \end{aligned}$$

Proof of Proposition 1. Consider $\gamma \in [0, \gamma')$, so that $\kappa^t = 1$ and $\eta^t = 0$ hold sufficiently near the steady state. The dynamics resembles that for the non-full-support steady state in the model without a denomination structure.

As a result, the Jacobian of the dynamical system is identical to that of Proposition 5 of [2] except that $\gamma > 0$ leads to a different steady state value w. By a similar computation, it is shown that out of the three eigenvalues of the Jacobian of (5)–(7), one eigenvalue which is from (5) is unity, another is smaller than one and the other is greater than one. Equation (5) implies unit-root convergence (see figure 2 in [2]) and the stable manifold is two dimensional.

When $\gamma \in (0, \gamma')$, the steady state w is an interior point of W. Then the standard approach (i.e., studying the dimension of the stable manifold) applies. The two-dimensional stable manifold and one initial condition imply local stability and indeterminacy.

When $\gamma = 0$, the value function is on the boundary of W because $w_1 = 0$. As Proposition 5 of [2] shows, this peculiarity does not go along with the unit-root of the law of motion, ruling out any such convergent paths to the steady state. In this case, the steady state is unstable. ■

Proof of Proposition 2. The existence of the steady states is shown by a guess and verify process. We impose strategies in Tables 1 and 2 on (5)–(7) and show that the resulting (π, w) is actually consistent with the optimality conditions for the strategies:

$$
\begin{array}{rrcl}
(\$1, \$1)\text{-meetings} & u(\beta w_2 - \beta w_1) & > & \beta w_1 \qquad (8) \\
(\$1, 0)\text{-meetings} & u(\beta w_1) & > & \beta w_1 \qquad (9) \\
(\$2, \$1)\text{-meetings} & u(\beta w_2 - \beta w_1) & > & \beta w_2 - \beta w_1 \qquad (10) \\
2\ \$1\text{s or paying } \$2 & \dfrac{\pi_0}{n}\left[u(\beta w_1) + \beta w_1\right] - \gamma & \geq & \dfrac{\pi_0}{n} u(\beta w_2) \qquad (11) \\
2\ \$1\text{s or keeping } \$2 & \dfrac{\pi_0}{n}\left[u(\beta w_1) + \beta w_1\right] - \gamma & > & \dfrac{\pi_0}{n} \beta w_2. \qquad (12)
\end{array}
$$

Inequalities (11) (resp. (12)) means that choosing to carry 2 \$1 bills and offering only \$1 in (2\$1s, 0)-meetings is at least as good as (resp. is better than) choosing to carry one \$2 bill and offering (resp. reserving) \$2 in (\$2, 0)-meetings. Equation (11) must hold with equality for the mixed-strategy steady state and with strict inequality for the pure-strategy steady state. The existence proof is similar to that in [2]. It consists of three lemmas and is provided in the appendix. Strict inequalities are important for the following stability analysis, because it guarantees that the inequalities also hold in the vicinity of the steady state.

The stability analysis for the pure-strategy steady state is standard. We derive the 3 by 3 Jacobian for the difference equation system (5)–(7) that is evaluated at the steady state. Then we show that one eigenvalue which is from (5) is smaller than one and the other two eigenvalues are greater than one, so that the stable manifold is one-dimensional.[7] This shows the local stability and determinacy of the pure-strategy full-support steady state.

The stability of the mixed-strategy full-support steady state is two-folds. For the model without denomination or carrying cost, [2] first showed that if the initial distribution is close to that of the mixed-strategy full-support steady state, the economy can reach the steady state in one period by people coordinating in the initial randomization. It is straightforward that the same logic applies also to our setting with carrying cost. Here, we show that such 'jump' is the unique convergent path by ruling out the possibility of a gradual convergence. For that purpose we show that when the distribution gradually goes to that of the steady state, (11) generically does not hold with equality.

Let $\Delta\eta(z)$, $\Delta\pi_1(z)$ and $\Delta w(z)$ be the z-transforms[8] of $\{\Delta\eta^t\}_0^\infty$, $\{\Delta\pi_1^t\}_0^\infty$ and $\{\Delta w^t\}_0^\infty$, respectively, where Δ's in the latter indicate the deviation from the steady state values. We denote by Φ and ϕ the r.h.s. of the law of motion (5) and Bellman equations (6)–(7), respectively, and let Φ_η, Φ_π, ϕ_π and ϕ_w denote their derivatives with respect to the subscript variable which is evaluated at the steady state. Linearizing equalities (11) and (5)–(7) with respect to (π_1^t, w^t, η^t) and applying the z-transforms, we get

$$[u'(\beta w_1)+1]\Delta w_1(z) = u'(\beta w_2)\Delta w_2(z) \tag{13}$$

$$\begin{pmatrix}\Delta\pi_1(z)\\ \Delta w(z)\end{pmatrix} = [Iz - A^1]^{-1}[\begin{pmatrix}\Phi_\eta\\ 0\end{pmatrix}\Delta\eta(z) + \begin{pmatrix}\Delta\pi_1^0\\ \Delta w^0\end{pmatrix}z], \tag{14}$$

where A^1 is the Jacobian of (32) with (π, w) being the mixed strategy steady state and $\zeta = 1$.
With (13), and multiplying (14) by $\begin{pmatrix} 0 & u'(\beta w_1)+1 & -u'(\beta w_2)\end{pmatrix}$, we get

$$\begin{aligned} &\begin{pmatrix} 0 & u'(\beta w_1)+1 & -u'(\beta w_2)\end{pmatrix}\operatorname{adj}[Iz - A][\begin{pmatrix}\Phi_\eta\\ 0\end{pmatrix}\Delta\eta(z) + \begin{pmatrix}\Delta\pi_1^0\\ \Delta w^0\end{pmatrix}z] \\ = \;& (z-\Phi_\pi)\left|Iz - \phi_w^{-1}\right|\{[u'(\beta w_1)+1]\Delta w_1^0 - u'(\beta w_2)\Delta w_2^0\}. \end{aligned} \tag{15}$$

[7] Please find the matrix computation in the appendix.

[8] The z-transform of a sequence of numbers $\{y_t\}$ is $Y(z) = \sum_{t=0}^\infty \frac{y_t}{z^t}$. Please refer to sections 8.2–8.4 in [4] for a detailed discussion.

The coefficient of $\Delta\eta(z)$ in the l.h.s. of (15), $\begin{pmatrix} u'(\beta w_1)+1 & -u'(\beta w_2) \end{pmatrix} \phi_\pi * (\Phi_\eta \, |\phi_w^{-1}|)$, is nonzero generically, because we can modify the shape of function u so that $u'(\beta w_1)$ and $u'(\beta w_2)$ can be changed arbitrarily without changing $u(\beta w_1)$ and $u(\beta w_2)$. (15) holds as an identity only if $\Delta\eta(z) = \Delta\eta^0$. Following (14), the unique path has $\binom{\Delta\pi_1^0}{\Delta w^0} = -A^{-1}\binom{\Phi_\eta}{0}\Delta\eta^0$, with $\Delta\pi_1^0$ given by the initial condition, and $\Delta\eta^t = \Delta\pi_1^t = \Delta w_1^t = \Delta w_2^t = 0$ for all $t \geq 1$. This rules out a gradual convergence to the mixed-strategy steady state. ■

6 Appendix

Supplementary materials for the detailed proof are presented. It will be convenient to consider the following form of conditions (11) and (12):

$$\text{Portfolio stage} \qquad u(\beta w_1) + \beta w_1 - \frac{n\gamma}{\pi_0} \geq u(\beta w_2) \tag{16}$$

$$\text{2 \$1s vs. \$2} \qquad \& u(\beta w_1) + \beta w_1 - \frac{n\gamma}{\pi_0} > \beta w_2. \tag{17}$$

Lemma 1 *If a monetary full-support steady state exists for a sufficiently small $\gamma > 0$, then*

(i) (8)-(12) hold; and

(ii) π_1 satisfies $\pi_1 \leq \pi_1^* \equiv (\sqrt{1+12m(1-m)}-1)/3$, where the inequality is strict if and only if $\eta^t < 1$.

Proof. The proof studies the fixed point of (5)–(7).

(i) Being a monetary steady state implies $w_2 > 0$, and having a full-support distribution implies $\pi_1 > 0$. Then (6) implies $w_1 > 0$ for a sufficiently small γ. We call the weak inequality version of (8)–(12), where the only change made in (8)–(12) is just replacing the strong inequalities by weak ones, (8)–(12) at least weakly. Then we have the following.

Step 1: Any full-support monetary steady state satisfies (8)–(12) at least weakly for a sufficiently small γ.

Proof of Step 1

Recall that the probability of paying \$1 in the (\$1, \$1)-meetings is denoted by κ in the main text. First, we want to show $\kappa > 0$, and that (8) and (16)-(17) hold at least weakly.

Suppose by way of contradicition both (8) and (9) hold with a reversed weak inequality. We can derive from (6) that $w_1 < 0$, a contradiction to $w_1 > 0$.

Suppose by contradiction that (9) and the reversed strict inequality in (8) hold. These imply

$$\beta w_2 - \beta w_1 < \beta w_1. \tag{18}$$

Note that (9) implies $0 < \beta w_1 < \bar{x}$, with $\bar{x}$ being the positive fixed point of $u(x)$. Thus we have $0 \leq \beta w_2 - \beta w_1 < \bar{x}$, which in turn implies (10) at least weakly. This weak inequality and applying utility function $u(\cdot)$ on both sides of (18) give (17) for a sufficiently small γ. Because u is strictly concave and (8) does not hold, we have $u(\beta w_2) - u(\beta w_1) < \beta w_1$ and hence (16) for a sufficiently small γ. Therefore, an agent with wealth 2 chooses two \$1 bills at the portfolio stage ($\eta = 1$) for a sufficiently small γ. For π_1 to be strictly positive in (5), we must have $\kappa > 0$ and hence the weak (8), a contradiction.

In summary, (8) holds at least weakly.

Suppose by contradiction that either (11) or (12) holds with reversed strict inequality, or $\eta = 0$ equivalently. Then (5) implies $\pi_1 = 0$, a contradiction to being a full-support steady state.

The weak (8) implies $\beta w_2 - \beta w_1 > \beta w_1$. Combining this with the weak (17) gives $u(\beta w_1) - \frac{n\gamma}{\pi_0} > \beta w_1$ and hence (9) for a sufficiently small γ.

Suppose now by way of contradiction that (10) does not hold even weakly: $u(\beta w_2 - \beta w_1) < \beta w_2 - \beta w_1$. Then the weak (17) for a sufficiently small γ implies $\beta w_2 - \beta w_1 < \beta w_1$. But the weak (8) and supposition imply $\beta w_2 - \beta w_1 > \beta w_1$, which is a contradiction. (End of proof of Step 1)

Step 2: If (8)–(12) hold weakly, then (8)–(10) and (12) hold strictly.

Proof of Step 2

When (8)–(12) hold at least weakly, we can eliminate 'max' operators from (6)–(7). Then subtracting (6) from (7) gives

$$w_2 - w_1 = \frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta} w_1 - \frac{n}{n(1-\beta)+(1-\pi_2)\beta}\gamma, \tag{19}$$

and βw_1 satisfies

$$\begin{aligned} & \tfrac{\beta}{n(1-\beta)+(1-\pi_2)\beta}\left[\pi_0 u(\beta w_1) + \pi_1 u\left(\tfrac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1 - \tfrac{n\beta}{n(1-\beta)+(1-\pi_2)\beta}\gamma\right) - n\gamma\right] \\ = & \qquad \beta w_1. \end{aligned} \tag{20}$$

Suppose by way of contradiction that (8) does not hold. Then, we have

$$\begin{aligned}
\beta w_1 &\leq \frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}(u(\beta w_1)-\frac{n\gamma}{\pi_0}) \\
&< u\left(\frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}\beta w_1\right)+\frac{n\gamma\beta}{n(1-\beta)+\pi_0\beta} \\
&< u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right)+\frac{n\gamma\beta}{n(1-\beta)+\pi_0\beta} \\
&= u(\beta w_2-\beta w_1+\frac{n\beta}{n(1-\beta)+(1-\pi_2)\beta}\gamma)+\frac{n\gamma\beta}{n(1-\beta)+\pi_0\beta},
\end{aligned}$$

where the first inequality is by substituting the supposition into (20) and the second is by Jensen's inequality and strict concavity of u. For a sufficiently small γ, the above implies that (8) hold.

Inequalities $u(\beta w_1) > u(\beta w_2 - \beta w_1) > \beta w_1 > \beta w_2 - \beta w_1$, where the first and the third inequalities are by (19) and the second is (8), implies that inequality (12) holds for a sufficiently small γ.

Suppose by way of contradiction that (9) does not hold: $u(\beta w_1) \leq \beta w_1$. Then (8) implies $\beta w_2 - \beta w_1 > \beta w_1$. Combining this with (17) gives $u(\beta w_1) > \beta w_1$, which is a contradiction.

Suppose now by way of contradiction that (10) does not hold: $u(\beta w_2 - \beta w_1) \leq \beta w_2 - \beta w_1$. Then (17) implies $\beta w_2 - \beta w_1 \leq \beta w_1$. But (8) and supposition imply $\beta w_2 - \beta w_1 > \beta w_1$, which is a contradiction.

In summary, (8)–(10) and (12) hold strictly. (End of proof of Step 2)

(ii) Letting $\kappa = 1$ in (5) and solving it for π_1 yields

$$\pi_1 = \sqrt{\left(\frac{\eta}{4-\eta}\right)^2 + 4m(1-m)\frac{\eta}{4-\eta}} - \frac{\eta}{4-\eta}.$$

Here $\pi_1 \in [0, \pi_1^*]$ is strictly increasing in $\eta \in [0,1]$ and is equal to π_1^* iff $\eta = 1$. ■

Lemma 2 *A monetary full-support steady state exists for sufficiently small γ if and only if there exists $(\pi_1, x) \gg 0$ such that*

$$x = \frac{\delta}{1-\pi_2}\left[\pi_0 u(x) + \pi_1 u(\delta x - \frac{\delta n\gamma}{(1-\pi_2)\beta}) - n\gamma\right] \equiv h(x, \pi_1, \gamma) \qquad (21)$$

and

$$u[(1+\delta)x - \frac{\delta n\gamma}{(1-\pi_2)\beta}] \leq u(x) + x - \frac{n\gamma}{\pi_0}, \tag{22}$$

where

$$\delta = \frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta} < 1 \tag{23}$$

and where (22) must hold with equality if $\pi_1 < \pi_1^*$.

Proof. (Necessity) By lemma 1, inequalities (8)-(12) hold for any full-support steady state, if γ is sufficiently small. Under these inqualities, the Bellman equations (6)–(7) become (21) and (23) with $x = \beta w_1$ and $(1+\delta)x - \frac{\delta n\gamma}{(1-\pi_2)\beta} = \beta w_2$. Also, (16) implies (22).
(Sufficiency) The proof resembles a guess and verify argument. Suppose we have such (π_1, x). Let $\beta w_1 = x$ and $\beta w_2 = (1+\delta)x - \frac{\delta n\gamma}{(1-\pi_2)\beta}$. Then we have (19)-(20). The same arguments as step 2 of the proof of lemma 1 show that (8)–(10) and (12) hold. Also, (11) is given by (22), Therefore we have (8)-(12), where (11) holds with equality if and only if (22) holds with equality. When $\pi_1 = \pi_1^*$, let $\eta = 1$, and the law of motion (5) is satisfied. When $\pi_1 < \pi_1^*$, then (22) holds with equality, and any η can be the optimal. We choose the unique one that solves (5). In summary, lemma-1 trade is the optimal.

Under such strategy, the Bellman equation (6)–(7) is equivalent to (21)–(23). ■

Lemma 3 *A monetary full-support steady state exists for a sufficiently small* $\gamma > 0$ *if and only if* $u'(0) > \frac{n(1-\beta)}{\beta(1-m)} + 1$.

Proof. In this proof, we denote δ as δ_{π_1} to make the dependence on π_1 explicit. First we show necessity. By lemma 1, the existence of a full-support steady state implies (8)-(12). Then, the Bellman equations (6)–(7) imply (21) with $x = \beta w_1$. Because the r.h.s. of (21) is concave in x, to have a positive solution for a sufficiently small γ, we must have

$$h_1(0, \pi_1, 0) = \delta_{\pi_1}\left[\frac{\pi_0}{1-\pi_2} + \frac{\pi_1\delta_{\pi_1}}{1-\pi_2}\right]u'(0) \equiv J_{\pi_1}u'(0) > 1, \tag{24}$$

or equivalently $u'(0) > 1/J_{\pi_1}$. Some algebra gives

$$\begin{aligned} 1/J_{\pi_1} &= \frac{n(1-\beta)}{\beta(1-m)} + 1 + \frac{n(1-\beta)}{\beta} \cdot \frac{\pi_1 n(1-\beta) + \beta\pi_1\pi_0}{[\pi_0 n(1-\beta) + \beta(1-\pi_2)^2](2-2m)} \\ &\geq \frac{n(1-\beta)}{\beta(1-m)} + 1, \end{aligned} \tag{25}$$

which completes the argument for necessity.

Next, we turn to sufficiency. Notice that (25) holds with equality if and only if $\pi_1 = 0$. Then, the assumption implies $h_1(0,0,0) = u'(0)J_0 > 1$. Equivalently the equation $h(x,0,\gamma) = x$ has two solutions for any sufficiently small γ. Because $h(x,0,\gamma)$ is concave in x, we have $h_1(x_0,0,\gamma) = \delta_0 u'(x_0) < 1$ with x_0 being the larger solution.

When $\pi_1 = 0$, we have

$$\begin{aligned} u[(1+\delta_0)x_0 - \frac{\delta_0 n\gamma}{(1-\pi_2)\beta}] - u(x_0) + \frac{n\gamma}{\pi_0} &< u^{'}(x_0)\delta_0 x_0 - \frac{u^{'}(x_0)\delta_0 n\gamma}{(1-\pi_2)\beta} + \frac{n\gamma}{\pi_0} \\ &< x_0 - \frac{n\gamma}{\pi_0\beta} + \frac{n\gamma}{\pi_0} \\ &< x_0, \end{aligned} \tag{26}$$

where the first inequality is by concavity of $u(\cdot)$, and where the second is by $\delta_0 u'(x_0) < 1$. The following are divided into two cases.

Case 1: There exists $\bar{\pi}_1 \in (0, \pi_1^*)$ such that $h_1(0,\bar{\pi}_1,0) = 1$ and $h_1(0,\pi_1,0) > 1$ for all $\pi_1 \in (0,\bar{\pi}_1)$. Then, for any sufficiently small γ, (21) has two positive solutions for all $\pi_1 \in (0,\bar{\pi}_1)$. View the larger one x_{π_1} as a function of π_1.

Then, we have

$$\begin{aligned} & \lim_{\pi_1\to\bar{\pi}_1}\lim_{\gamma\to 0} \frac{u(x_{\pi_1}) + x_{\pi_1} - \frac{n\gamma}{\pi_0} - u((1+\delta_{\pi_1})x_{\pi_1} - \frac{\delta_{\pi_1} n\gamma}{(1-\pi_2)\beta})}{x_{\pi_1}} \\ = & \lim_{\pi_1\to\bar{\pi}_1} \frac{u(x_{\pi_1}) + x_{\pi_1} - u((1+\delta_{\pi_1})x_{\pi_1})}{x_{\pi_1}} \\ < & \lim_{\pi_1\to\bar{\pi}_1} \frac{x_{\pi_1} - u'((1+\delta_{\pi_1})x_{\pi_1})\delta_{\pi_1}x_{\pi_1}}{x_{\pi_1}} \\ = & 1 - u'(0)\delta_{\bar{\pi}_1} \\ = & \left[\frac{\bar{\pi}_1(\delta_{\bar{\pi}_1} - 1)}{1-\bar{\pi}_2}\right] u'(0)\delta_{\bar{\pi}_1} \\ < & 0. \end{aligned} \tag{27}$$

The first inequality follows from concavity of u. The second equality uses the fact that $x_{\pi_1} \to 0$ as $\pi_1 \to \bar{\pi}_1$ and $\gamma \to 0$. The last equality uses $1 = h_1(0, \bar{\pi}_1, 0)$. The last inequality is by $\delta_{\bar{\pi}_1} < 1$.

With the limiting condition of (27), we can find a π_1 sufficiently close to $\bar{\pi}_1$ such that (22) holds with reversed strict inequality for any sufficiently small γ. As π_1 increases in $[0, \bar{\pi}_1]$, the inequality in (22) switches directions, and the intermediate value theorem implies the existence of $\hat{\pi}_1 \in (0, \bar{\pi}_1)$ such that

$$u(x_{\hat{\pi}_1}) + x_{\hat{\pi}_1} - \frac{n\gamma}{\hat{\pi}_0} = u((1 + \delta_{\hat{\pi}_1})x_{\hat{\pi}_1} - \frac{\delta_{\hat{\pi}_1} n\gamma}{(1 - \hat{\pi}_2)\beta}). \tag{28}$$

By lemma 2, such a pair $(\hat{\pi}_1, x_{\hat{\pi}_1})$ forms a mixed-strategy full-support steady state.

Case 2: $h_1(0, \pi_1, 0) > 1$ for all $\pi_1 \in [0, \pi_1^*]$. As in Case 1, view the larger solution to (21), x_{π_1}, as a function of π_1. If (22) holds with reversed inequality at π_1^*, then the intermediate value theorem and lemma 2 implies the pair $(\hat{\pi}_1, x_{\hat{\pi}_1})$ forms a mixed-strategy full-support steady state; Otherwise, lemma 2 implies that there is a (pure-strategy) full-support steady state.

Lemma 1 rules out other possible full-support steady states. ■

The stability analysis considers the following matrices:

$$\Psi_\pi^\zeta = 1 - \frac{\sqrt{1 + 12m(1-m)}}{n}\zeta, \tag{29}$$

$$\phi_\pi^\zeta = \begin{bmatrix} \frac{-\beta w_1 - u(\beta w_1) + 2u(\beta \Delta w)}{2n} \\ \frac{-\beta w_2 - [\zeta\{u(\beta w_1) + \beta w_1\} + (1-\zeta)u(\beta w_2)] + 2[u(\beta \Delta w) + \beta w_1]}{2n} \end{bmatrix}, \tag{30}$$

$$\phi_w^\zeta = \begin{pmatrix} \frac{(n-1+\pi_2)\beta}{n} + \frac{\pi_0 \beta u'(\beta w_1)}{n} - \frac{\pi_1 \beta u'(\beta \Delta w)}{n} & \frac{\pi_1 \beta u'(\beta \Delta w)}{n} \\ \frac{\zeta \pi_0 (u'(\beta w_1) + 1)\beta}{n} + \frac{\pi_1 (1 - u'(\beta \Delta w))\beta}{n} & \frac{(n-1+\pi_2)\beta}{n} + \frac{\pi_0 (1-\zeta)\beta u'(\beta w_2)}{n} + \frac{\pi_1 \beta u'(\beta \Delta w)}{n} \end{pmatrix}, \tag{31}$$

and

$$A^\zeta \equiv \begin{bmatrix} \Psi_\pi^\zeta & O \\ -(\phi_w^\zeta)^{-1}\phi_\pi^\zeta & (\phi_w^\zeta)^{-1} \end{bmatrix}, \tag{32}$$

where $\Delta w \equiv w_2 - w_1$ and $\zeta \in \{0, 1\}$. The following contains the existence argument and the matrix computation in the Proof of Proposition 2

Remaining Proof of Proposition 2. When β is sufficiently close to one, the pure-strategy full-support steady state exists for any sufficiently small γ. To see this, let $\pi_1 = \pi_1^*$, and take the limiting process of first $\gamma \to 0$

and then $\beta \to 1$. Conditions (21), (22) and (23) approach $x_{\pi_1^*} = u(x_{\pi_1^*})$, $u(2x_{\pi_1^*}) < u(x_{\pi_1^*}) + x_{\pi_1^*}$ and $\delta_{\pi_1^*} = 1$, respectively. By strict concavity of u, the limiting condition of (22) holds. Thus, (22) holds, and hence the pure-strategy full-support steady state exists for (β, γ) sufficiently close to $(1, 0)$.

The proof of lemma 3 implies that a mixed-strategy full-support steady state exists for any sufficiently small γ if a pure-strategy one does not, and that $u'(0) > 1/J_{\pi_1^*}$ is necessary for the existence of a pure-strategy full-support steady state. By lemma 3, if $u'(0) \in (\frac{n(1-\beta)}{\beta(1-m)} + 1, 1/J_{\pi_1^*})$[9], then a mixed-strategy full-support steady state exists for any sufficiently small γ, which implies the generic existence.

Turn to stability of the pure-strategy full-support steady state. Trading one unit in all trade meetings is a strictly preferred strategy at the steady state, so it is also optimal in its neighborhood. The Jacobian of this steady state at the steady state by (π^*, w^*) is (32) with $\zeta = 1$. Because the top-right submatrix is a zero matrix, one eigenvalue is given by (29), which is smaller than one, and the other two are the reciprocals of eigenvalues of ϕ_w^γ. In what follows, we show that eigenvalues of ϕ_w^γ are smaller than one in absolute value.

Because $h(\beta w_1, \pi_1^*, \gamma)$ is concave in w_1, and hence $h_1(\beta w_1^*, \pi_1^*, \gamma) < 1$, we have

$$\frac{n(1-\beta) + (1-\pi_2^*)\beta}{\beta} > \pi_0^* u'(\beta w_1^*) + \pi_1^* \frac{(1-\pi_2^*)\beta}{n(1-\beta) + (1-\pi_2^*)\beta} u'(\beta \Delta w^*). \tag{33}$$

The eigenvalues of a general 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are given by

$$\eta_+, \eta_- = \frac{a + d \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

[9] (25) implies this set is nonempty. The condition is satisfied for β of intermediate value.

Because

$$
\begin{aligned}
& (a-d)^2+4bc \\
= & \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-2\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right]^2 \\
& +4\left[\frac{1-\pi_2^*}{n}\beta+\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right]\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) \\
= & \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)\right]^2+4\frac{1-\pi_2^*}{n}\beta\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)>0,
\end{aligned}
$$

both eigenvalues are real. They are smaller than one in absolute value if and only if $a+d<2$ and $(1-a)(1-d)-bc>0$. Some algebra gives

$$
\begin{aligned}
& 1-a+1-d \\
= & \ 2\left(1-\frac{n-1+\pi_2^*}{n}\beta\right)-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*) \\
> & \ 2\frac{n(1-\beta)+(1-\pi_2^*)\beta}{n}-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}\beta u'\left(\beta\Delta w^*\right) \\
> & \ \frac{n(1-\beta)+(1-\pi_2^*)\beta}{n};\ \text{and}
\end{aligned}
$$

$$
\begin{aligned}
& (1-a)(1-d)-bc \\
= & \left(1-\frac{n-1+\pi_2^*}{n}\beta-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)+\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right)\left(1-\frac{n-1+\pi_2^*}{n}\beta-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right) \\
& -\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\left[\frac{1-\pi_2^*}{n}\beta+\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right] \\
= & \frac{(n(1-\beta)+(1-\pi_2^*)\beta)\beta}{n^2}\times \\
& \left(\frac{n(1-\beta)+(1-\pi_2^*)\beta}{\beta}-\pi_0^*u'(\beta w_1^*)-\pi_1^*\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}u'(\beta\Delta w^*)\right) \\
> & \ 0,
\end{aligned}
$$

where the last inequalities of the above two conditions follow from (33). In summary, the Jacobian has only on eigenvalue smaller than one in absolute value. The pure-strategy full-support steady state has a one-dimensional stable manifold. Because the convergent path is restricted by one initial condition, this full-support steady state is locally stable and determinate. ■

Matrix Computation in Prop 1. The Jacobian (32) with $\zeta = 0$ reduces to

$$A^{\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ -r/a' & 1/a' & 0 \\ -s/d' & 0 & 1/d' \end{bmatrix}, \tag{34}$$

where $r \equiv \frac{1}{n}u(\beta w_2)$, $s \equiv \frac{1}{2n}[u(\beta w_2) - \beta w_2] > 0$, $a' \equiv \frac{(n-1+m)\beta}{n} + \frac{1-m}{n}\beta u'(0)$, and $d' \equiv \frac{(n-1+m)\beta}{n} + \frac{1-m}{n}u'(\beta w_2)\beta$. Note that because w_2 is the larger positive solution to (3), $a' > 1$ and $d' \in (0, 1)$ hold. Eigenvalues of (34) are simply its diagonal elements.

The law of motion has unit-root convergence. The associated eigenvector, which constitutes a base of the eigenspace, has the form

$$\begin{bmatrix} 1 \\ \frac{-r}{a'-1} \\ \frac{s}{1-d'} \end{bmatrix}.$$

■

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